11.1 Introduction

Control charts, developed using quality characteristics, expressed in absolute quantitative terms, or numerical values are called *Variables Control charts*.

Examples:

1) length, diameter, weight, etc. of a product,
2) temperature of a process, and so on.

Process often requires controlling –

- mean or average or central tendency ($\overline{X}, or\ \mu$), and
- dispersion or variability or standard deviation (σ or s), or range (R)

of a quality characteristic.

Commonly used variable control charts are:

1. $\overline{X}-R$ Chart
2. $\overline{X}-S$ Chart
3. Moving Range (MR) chart
4. Cumulative Sum (CUSUM) chart
5. Exponentially Weighted Moving Average (EWMA) chart

11.2 $\overline{X} - R$ Chart

Suppose that the length of a product is to be controlled.

Sample of size ***n*** is randomly taken from the products, in a certain time period (say, a day).

Mean or average of that sample is calculated as follows:

$$\overline{X}_i = \frac{\sum_{j=1}^{n} X_j}{n}$$ ....eq. (1) where, $\overline{X}_i$ is the sample mean, taken in a day.

Such samples, with sufficiently large size (to justify Normal Distribution), are taken for **m** days in a month.

In this case, mean of the sampling distribution of *m* numbers of sample means is –

$$\overline{\overline{X}} = \frac{\sum_{i=1}^{m} \overline{X}_i}{m}$$ ....eq. (2)

Range of sample *i* is - $R_i = X_{max} - X_{min}$

Mean value of range for total m samples is = $\overline{R} = \frac{\sum_{i=1}^{m} R_i}{m}$ ....eq. (4)

The central limit theorem states that –

$\bar{\bar{X}} \approx \mu$ ...where, $\bar{\bar{X}}$ is the "mean of sample means" and μ is the population mean.

But, the standard deviation of the sampling distribution of sample means is –

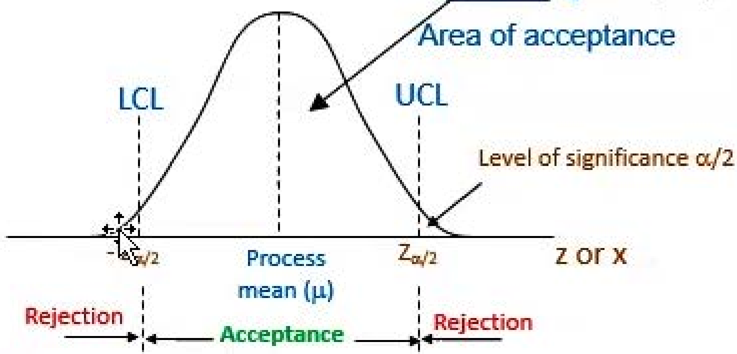
$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

[**Note:** statistical distribution of $\bar{X}$ values is called "Sampling distribution of sample means", which is approximately a Normal Distribution.]

For a two-tail test:

$$P = \mu - Z_{\alpha/2}\sigma_{\bar{X}} \le \bar{X} \le \mu + Z_{\alpha/2}\sigma_{\bar{X}}) = \boxed{1-\alpha}$$

where α is the level of significance and $(1-\alpha)$ is the level of confidence



$$\left.\begin{aligned} UCL &= \mu + Z_{\alpha/2}\sigma_{\bar{X}} = \mu + \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} = \bar{\bar{X}} + \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} \\ CL &= \mu = \bar{\bar{X}} \\ LCL &= \mu - Z_{\alpha/2}\sigma_{\bar{X}} = \mu - \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} = \bar{\bar{X}} - \frac{Z_{\alpha/2}\sigma}{\sqrt{n}} \end{aligned}\right\}$$

.....Eq. (3)

Often, population s.d. σ is not known. In that case, an estimate of σ is taken as $\hat{\sigma}$ which is basically mean value or "Expected value" of σ –

$$\hat{\sigma} = E(\sigma) = \frac{\bar{R}}{d_2}$$

.....Eq. (6)....value of d_2 parameter can be found from Table B (p.330)

Through some computations and assumptions, we get (from Eq. 3) limits of $\bar{X}$ chart –

$$UCL_{\bar{X}} = \bar{\bar{X}} + A_2\bar{R} \qquad CL_{\bar{X}} = \bar{\bar{X}} \qquad LCL_{\bar{X}} = \bar{\bar{X}} - A_2\bar{R}$$

.........Eq. (8)

Similarly, we compute the limits of *R* chart –

$$UCL_R = \bar{R}D_4 \qquad CL_R = \bar{R} \qquad LCL_R = \bar{R}D_3$$

.........Eq. (10)

It is important to mention that the *R* chart is developed first. If the process is found in-control, only then $\bar{X}$ chart should be constructed and analyzed.

Example 1: $\bar{X}$– *R Chart*

Metlab Casting Company Ltd. produces steel pipes of a certain diameter, considered as a critical quality characteristic. The company decided to use $\bar{X} - R$ chart to control diameter.

From a day's production, a sample of 5 pipes is selected randomly from the production line and their diameters are recorded. The average diameter and range of this sample (of size 5) are computed and recorded in a table (Table 11.1). The inspector collected this type of samples in 22 working days in the month of February. Thus, $\bar{X}$ and R values of 22 samples are recorded in the table. The next step for the company is to develop trial control limits.

Day	$\bar{X}$	R	Day	$\bar{X}$	R
1	10.724	0.040	12	10.730	0.026
2	10.730	0.016	13	10.735	0.028
3	10.718	0.040	14	10.726	0.041
4	10.728	0.014	15	10.724	0.025
5	10.730	0.027	16	10.720	0.017
6	10.720	0.020	17	10.727	0.035
7	10.720	0.038	18	10.720	0.037
8	10.711	0.026	19	10.726	0.030
9	10.713	0.027	20	10.724	0.012
10	10.718	0.008	21	10.718	0.030
11	10.717	0.039	22	10.722	0.012
			Total:	235.901	0.608

Table 11.1: $\bar{X}$ and R Values for steel pipe diameters (in centimeters).

$$\bar{R} = \frac{\sum_{i=1}^{m} R_i}{m} = \frac{0.608}{22} = 0.028$$

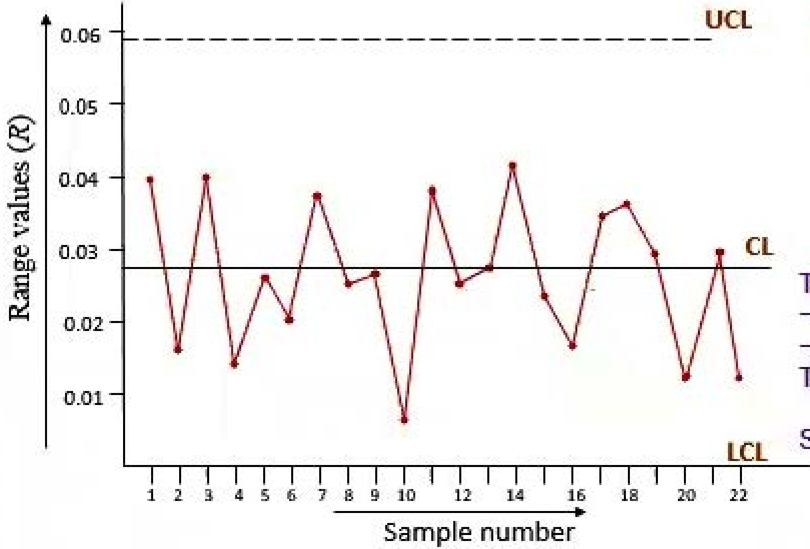
For samples of size $n = 5$,
Table B provides : $D_4 = 2.114$ and $D_3 = 0$.

From equation (10), the control limits for Range (R) chart are found –

$$UCL_R = \bar{R} D_4 = (0.028)(2.114) = 0.059$$
$$CL_R = \bar{R} = 0.028$$
$$LCL_R = \bar{R}\ D_3 = (0.028)(0)=0$$

An R chart is constructed using these limits and R values (from Table 11.1) are plotted, as shown in Figure 11.2 (next slide).

Figure 11.2: *R* chart



The plots are –
– pretty random around the mean, and
– within the limits.
Thus, in-control.

So, $\overline{X}$ chart can be plotted now.

$$\overline{\overline{X}} = \frac{\sum_{i=1}^{22} \overline{X}_i}{22} = \frac{235.901}{22} = 10.722$$

For samples of size n = 5, Table B provides : A_2 = 0.577,

$$\begin{aligned} UCL_{\overline{X}} &= \overline{\overline{X}} + A_2\overline{R} = 10.722 + 0.577(0.028) = 10.738 \\ CL_{\overline{X}} &= 10.722 \\ LCL_{\overline{X}} &= \overline{\overline{X}} - A_2\overline{R} = 10.722 - 0.577(0.028) = 10.706 \end{aligned}$$

An $\overline{X}$ chart is constructed using these limits, and $\overline{X}$ values (from Table 11.1) are plotted, as shown in Figure 11.3 (next slide).

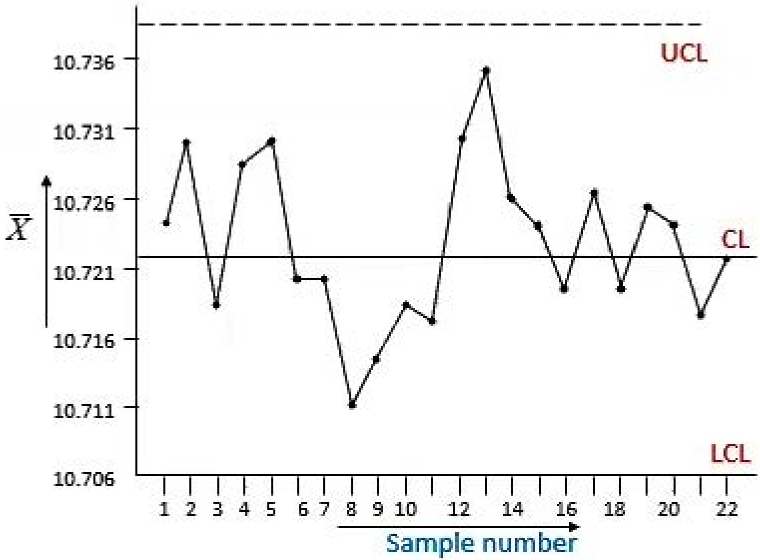


Figure 11.3: $\overline{X}$ Control chart.

The plots are –

- pretty random around the mean, and
- within the limits.

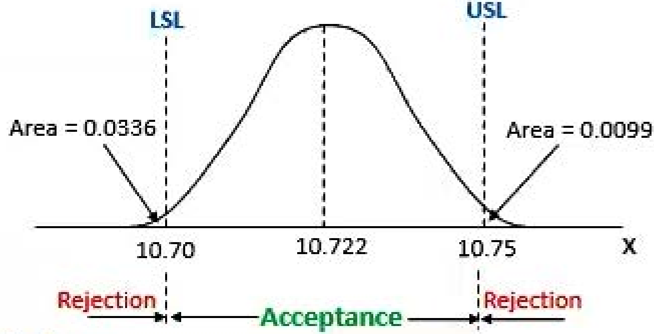
Thus, in-control.

So, we conclude that the process is fairly in-control.

Process Capability Analysis

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{0.028}{2.326} = 0.012$$

Suppose that the stated speciation limits are : 10.70 cm, 10.75 cm, as illustrated in the Figure (Figure 11.4).



Fraction non-conforming at the right and left tails are:

$$P(X > USL) = P\left(Z > \left[\frac{10.75 - 10.722}{0.012}\right]\right) = P(Z > 2.33) = 0.0099 = 0.99\%$$

$$P(X < LSL) = P\left(Z < \left[\frac{10.70 - 10.722}{0.012}\right]\right) = P(Z < -1.83) = 0.0336 = 3.36\%$$

Total fraction of 'products not meeting specifications' is : 0.0099 + 0.0336 = 0.0435 or 4.35%, which is quite high for a good process.

Process capability ratio, where σ is estimated by $\hat{\sigma}$, is –

$$PCR = \frac{USL - LSL}{6\hat{\sigma}} = \frac{10.75 - 10.70}{6 \times 0.012} = 0.694 < 1$$ ……**Process Potential Index C_p**

PCR value is alarmingly less than 1, which indicates that a large number of products will be nonconforming.

One important point to notice that proportions of products not meeting specifications on two sides (or the tails) of the normal distribution are not equal which further reveals marginal shifting of the process mean.

11.3 $\bar{X}$−S chart

Range (R) is not a good measure of variability, although it is easy to compute, as far as computational complexity is concerned.

Standard deviation is a better measure in all respects.

Thus, $\bar{X}$−S chart is better than $\bar{X}$−R chart, where S is sample standard deviation.

When number of data elements, i.e. the sample size n, increases further, say $n > 10$, range chart looses its credibility completely.

However, S chart is suggested, because in majority of the cases, population standard deviation σ is unknown.

11.3.1 The concepts of $\overline{X} - S$ chart

Sample standard deviation is -

$$S_i = \sqrt{\frac{\sum_{j=1}^{n}(x_j - \bar{x})^2}{n-1}}$$ .....for Sample of size n

Mean value of sample s.d. of *m* samples is – $\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$

The control limits for $\overline{X}$ chart are –

$$\left.\begin{aligned} UCL_{\overline{X}} &= \overline{\overline{X}} + A_3\bar{S} \\ CL_{\overline{X}} &= \overline{\overline{X}} \\ LCL_{\overline{X}} &= \overline{\overline{X}} - A_3\bar{S} \end{aligned}\right\}$$ ....Eq. (11)

The control limits for S chart are –

$$\left.\begin{aligned} UCL_S &= B_4\bar{S} \\ CL_S &= \bar{S} \\ LCL_S &= B_3\bar{S} \end{aligned}\right\}$$ ....Eq. (12)

Interpretation and analysis of this chart is also similar to $\overline{X}$- R chart.